MIXED FORMULATIONS AND HYBRID METHODS FOR HEAT AND MASS TRANSFER IN POROUS MEDIA

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Abstract
The present review illustrates part of the research work undertaken by the Laboratory of Transmission and Technology of Heat, LTTC, at POLI/COPPE/UFRJ, towards the construction of hybrid tools for the problem formulation, solution methodology, inverse problem analysis and computational implementation tasks inherent to the simulation process in thermal sciences and engineering. First, a problem reformulation strategy is discussed, known as the Coupled Integral Equations Approach (CIEA), which offers improved lumped-differential formulations in different classes of problems, in comparison against classical lumping schemes, allowing for a reduction on the number of independent variables to be considered in specific formulations. Second, the Generalized Integral Transform Technique (GITT) is reviewed, as a hybrid numerical-analytical solution methodology for diffusion and convection-diffusion problems. The relative merits of such approach over purely numerical procedures, in light of its hybrid nature, are then discussed. Examples are provided in each of these research fronts related to drying in capillary porous media as formulated by Luikov’s system of equations.

KEYWORDS
Hybrid methods, Lumped systems, Integral transforms, Luikov equations, Porous media, Drying.

INTRODUCTION
This lecture is the synthesis of a coordinated effort towards consolidating and constructing computational simulation tools based on hybrid numerical-analytical methodologies for multiphysics engineering problems, while promoting the advancement of research fronts for new technological frontiers that require the combination of different scientific and application areas.

Computer simulation of partial differential equations has nowadays a quite clear and definitive role in the analysis of engineering problems, within both the scientific and technological contexts. It is also now well understood that computer simulation is a key ingredient in narrowing scientific/technological gaps among different research and development groups. While commercial simulation packages are still far from perfect on cost-effectiveness, reliability, flexibility and accuracy, their presence in routine tasks of industry and consulting companies is an undeniable fact, which brings an increasing demand for improvements. It may appear at first that not much was left to be done on mathematical analysis and algorithm design for achieving simulation goals, and that the major enhancement still possible relies on the advancement of computer hardware. However, fundamental research on these aspects has barely started, along the last few decades, and the resulting enhancement can be orders of magnitude higher when mathematics and algorithms findings are combined with hardware improvement.

Based on such observation, our work at the Universidade Federal do Rio de Janeiro, Brasil, has been devoted to four interconnected research fronts aimed at concurrently enhancing the simulation process in thermal sciences and engineering. Following the physical model construction, hybrid tools for development of the mathematical model, of the solution methodology, of the inverse analysis, and of the algorithm implementation are investigated. The hybrid nature is present in all aspects of our work, including lumped-differential formulations, numerical-analytical methods, and symbolic-numerical computations. It should be added that a multi- and interdisciplinary motivation permeates the whole research, as the aim is to integrate various efforts that emanate from groups of different backgrounds, as well as to move forward towards multiple physical areas. In this sense, the new challenging problem areas encompass more than one physical subject, in connection with some of the most relevant technologies presently under development.
The present paper reviews part of this work and attempts to illustrate through examples some of our research achievements and difficulties. In the next section, a problem reformulation strategy is presented, known as the Coupled Integral Equations Approach (CIEA), which offers improved lumped-differential formulations in different classes of problems, in comparison with classical lumping schemes, allowing for a reduction on the number of independent variables to be considered in specific formulations, thus reducing simulation costs. Afterwards, the now well-established Generalized Integral Transform Technique (GITT) is reviewed, as a hybrid numerical-analytical solution methodology for diffusion and convection-diffusion problems. The relative merits of such approach over purely numerical procedures, in light of its hybrid nature, are also discussed, such as the automatic global accuracy control feature and the mild increase on computational costs for multidimensional nonlinear situations. The philosophy of aiming accuracy and its automatic control along the simulations, behind the last twenty years of developments, has been faithfully maintained. Thus, the concluded developments yield, first of all, benchmarking tools and results, which might also be used in conjunction with other numerical tools, either commercial or not, as a covalidation tool.

IMPROVED LUMPED-DIFFERENTIAL FORMULATIONS: THE COUPLED INTEGRAL EQUATIONS APPROACH (CIEA)

The solution of multidimensional convection-diffusion problems, in the realm of practical applications, presents difficulties associated with a marked analytical involvement and/or considerable computational effort. Considering these facts, it becomes of interest for the engineering practice, to propose simpler formulations of the original partial differential systems, through a reduction of the number of independent variables in the multidimensional situations, by integration (averaging) of the full partial differential equations in one or more space variables, but retaining some information in the direction integrated out, provided by the related boundary conditions. Different levels of approximation in such mixed lumped-differential formulations can be used, starting from the plain and classical lumped system analysis, towards improved formulations, obtained through Hermite-type approximations for integrals [1, 2]. Such approach has been already exploited in different heat and fluid flow problems [3-5], including phase change problems, extended surfaces, anisotropic heat conduction, heat exchangers analysis, Navier-Stokes equations and drying problems [6-15], and shall be illustrated within this section.

Hermite [1] developed a way of approximating an integral, based on the values of the integrand and its derivatives at the integration limits, in the form:

$$\int_{x_{i-1}}^{x_i} y(x)dx \approx \sum_{\nu=0}^{\alpha} C_{\alpha,\nu} y_{i-1}^{(\nu)} + \sum_{\nu=0}^{\beta} D_{\nu} y_i^{(\nu)}, \quad (1.a)$$

where $y(x)$ and its derivatives $y^{(\nu)}(x)$ are defined for all $x \in (x_{i-1}, x_i)$. Furthermore, it is assumed that the numerical values of $y^{(\nu)}(x_{i-1}) = y^{(\nu)}(x_i)$ for $\nu = 0, 1, 2, ..., \alpha$ and $y^{(\nu)}(x_i) \equiv y_i^{(\nu)}$ for $\nu = 0, 1, 2, ..., \beta$, are available at the end points of the interval.

In such a manner, the integral of $y(x)$ is expressed as a linear combination of $y(x_{i-1})$, $y(x_i)$ and their derivatives, $y^{(\nu)}(x_{i-1})$ up to order $\nu = \alpha$, and $y^{(\nu)}(x_i)$ up to order $\nu = \beta$. This is called the $H_{\alpha, \beta}$ approximation. A detailed derivation for arbitrary $\alpha$ and $\beta$ is presented in [2]. The resulting expression for the $H_{\alpha, \beta}$ approximation is given by:

$$\int_{x_{i-1}}^{x_i} y(x)dx = \sum_{\nu=0}^{\alpha} C_{\nu} (\alpha, \beta) h_{\nu}^{x_{i-1}} y_{i-1}^{(\nu)} + \sum_{\nu=0}^{\beta} C_{\nu} (\beta, \alpha) (-1)^{\nu} h_{\nu}^{x_i} y_i^{(\nu)} + O(h_{\nu}^{\alpha+\beta+3}), \quad (1.b)$$

where

$$h_{\nu} = x_i - x_{i-1} \quad ; \quad C_{\nu} (\alpha, \beta) = \frac{(\alpha + 1)! (\alpha + \beta + 1 - \nu)!}{(\nu + 1)! (\alpha - \nu)! (\alpha + \beta + 2)!}. \quad (1.c,d)$$

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In the following example, we consider just the two approximations, \( H_{0,0} \) and \( H_{1,1} \), given by:

\[
H_{0,0} \to \int_0^h y(x)dx \approx \frac{h}{2}(y(0) + y(h)), \quad (2)
\]

\[
H_{1,1} \to \int_0^h y(x)dx \approx \frac{h}{2}(y(0) + y(h)) + \frac{h^2}{12}(y'(0) - y'(h)), \quad (3)
\]

which correspond, respectively, to the well-known trapezoidal and corrected trapezoidal integration rules. The respective expressions for the errors in the approximations \( H_{0,0} \) and \( H_{1,1} \) are written as:

\[
E_{0,0} = -\frac{h^3}{12}y''(\eta), \quad \eta \in (0,h), \quad (4.a)
\]

\[
E_{1,1} = +\frac{h^5}{720}y''(\xi), \quad \xi \in (0,h) \quad (4.b)
\]

and can be employed to compose the final error expression in the desired averaged potential, which may then be bounded for a priori error analysis.

Under certain boundary conditions and other characteristics of the diffusion process, the fully differential formulations can be markedly simplified through a reduction of the number of independent variables involved. Thus, one or more space variables may be integrated out in the original differential formulations, yielding, through the use of expressions such as eqs. (2), (3), approximate formulations that retain local information on the remaining coordinates, and averaged information in the directions eliminated through integration.

**Application: Drying in Capillary Porous Media (CIEA)**

Despite all the progress achieved in the computational solution of drying problems formulated by the original set of Luikov's equations, these methodologies are still quite too complex for engineering-type work in the realm of applications. In particular due to the multidisciplinary aspects of this physical problem, appearing within various sciences branches where a profound mathematical background might not be a priori required, the development of simplified formulations becomes of major relevance. One such possibility of simplification is the classical lumped system analysis, based on the assumption of uniform distribution of the associated potentials over the whole problem domain or along selected coordinates. The alternative technique of producing approximate formulations described in this section, based on the use of Hermite-type approximations for integrals, is here examined. Therefore, the present example is aimed at illustrating improved lumped-differential formulations developed in the context of drying problems [3, 10, 13], starting from the Luikov system of partial differential equations [16, 17]. The integral transform solution of the original Luikov system provides the reference results [18-22] to illustrate the accuracy and applicability limits of the approximate formulations.

The physical problem under picture involves a cylindrical capillary porous medium of radius \( R_0 \) and length \( l \), initially at uniform temperature and uniform moisture content. One of the boundaries, which is impervious to moisture transfer, is put in contact with a heater. The other boundary is put in contact with the dry surrounding air, thus resulting in a convective boundary condition for both the temperature and the moisture content. The lateral surface of the cylinder is also supposed to be impervious to mass transfer, but heat losses at this boundary are taken into account through a convective boundary condition. The system of equations proposed by Luikov, for the modeling of such physical problem involving the drying of a capillary porous media, can be written in dimensionless form as [16, 17]:
\[
\frac{\partial \theta(R,Z,\tau)}{\partial \tau} = \alpha \frac{\partial^2 \theta(R,Z,\tau)}{\partial Z^2} - \beta \frac{\partial^2 \phi(R,Z,\tau)}{\partial Z^2} + \frac{r_\alpha^2}{R} \frac{\partial}{\partial R} \left[ R \frac{\partial \theta(R,Z,\tau)}{\partial R} \right] - \frac{r_\beta^2}{R} \frac{\partial}{\partial R} \left[ R \frac{\partial \phi(R,Z,\tau)}{\partial R} \right]
\]
\[
\frac{\partial \phi(R,Z,\tau)}{\partial \tau} = Lu \frac{\partial^2 \phi(R,Z,\tau)}{\partial Z^2} - LuPn \frac{\partial^2 \theta(R,Z,\tau)}{\partial Z^2} + \frac{r_\alpha^2}{R} \frac{\partial}{\partial R} \left[ R \frac{\partial \phi(R,Z,\tau)}{\partial R} \right] - \frac{r_\beta^2}{R} \frac{\partial}{\partial R} \left[ R \frac{\partial \theta(R,Z,\tau)}{\partial R} \right] + \frac{\alpha}{Lu} \frac{\partial}{\partial R} \left[ \frac{\partial \theta(R,Z,\tau)}{\partial R} \right] - \frac{\beta}{Lu} \frac{\partial}{\partial R} \left[ \frac{\partial \phi(R,Z,\tau)}{\partial R} \right] + R \frac{\partial}{\partial R} \left[ \frac{\partial \theta(R,Z,\tau)}{\partial R} \right] - \frac{R}{Lu} \frac{\partial}{\partial R} \left[ \frac{\partial \phi(R,Z,\tau)}{\partial R} \right] \quad \text{in } 0<\tau<1 \text{ and } 0<Z<1 \text{ for } \tau>0,
\]
\[
\theta(R,Z,0) = 0, \quad \phi(R,Z,0) = 0.
\]
\[
\frac{\partial \theta(0,Z,\tau)}{\partial R} = 0, \quad \frac{\partial \phi(0,Z,\tau)}{\partial R} = 0, \quad \text{at } R=0 \text{ and } Z=0 \text{ for } \tau>0.
\]
\[
\frac{\partial \theta(1,Z,\tau)}{\partial R} - Bi_\psi \left[ 1- \theta(1,Z,\tau) \right] = 0, \quad \text{at } R=1 \text{ and } 0<Z<1 \text{ for } \tau>0,
\]
\[
\frac{\partial \phi(1,Z,\tau)}{\partial R} = Pn \frac{\partial \theta(1,Z,\tau)}{\partial R}, \quad \text{at } R=1 \text{ and } 0<Z<1 \text{ for } \tau>0.
\]

The various dimensionless groups appearing above are defined as

\[
\theta(R,Z,\tau) = \frac{T(r,z,l)-T_s}{T_o-T_s}, \quad \phi(R,Z,\tau) = \frac{u_o-u(r,z,l)}{u_o}, \quad Q = \frac{q}{k(T_o-T_s)}, \quad \tau = \frac{at}{l^2}.
\]

\[
Lu = \frac{a_m}{a}, \quad Pn = \frac{T_o-T_s}{u_o-u_s}, \quad Bi_\psi = \frac{h_l}{k}, \quad Bi_m = \frac{h_m}{k_m}, \quad Ko = \frac{\lambda u_o-u_s}{c T_o-T_s}, \quad Bi_y = \frac{h_y R_y}{k}. \quad (6.a-d)
\]

\[
r_\alpha = \frac{l}{r}, \quad R = \frac{R}{R_0}, \quad Z = \frac{z}{l}, \quad Bi_m^* = Bi_m \left[ 1-(1-\varepsilon)Pn Ko Lu \right].
\]

\[
\alpha = 1 + \varepsilon KoLuPn, \quad \beta = \varepsilon KoLu,
\]

where \(a\) is the thermal diffusivity of the porous medium, \(a_m\) is the moisture diffusivity in the porous medium, \(c\) is the specific heat of porous medium, \(h_\alpha\) and \(h_\beta\) are the heat transfer coefficients at the top and lateral surfaces, respectively, \(h_m\) is the mass transfer coefficient, \(k\) is the thermal conductivity, \(k_m\) is the moisture conductivity, \(l\) is the thickness of porous medium, \(q\) is the prescribed heat flux, \(\lambda\) is the latent heat of evaporation of water, \(T_o\) is the temperature of the surrounding air, \(T_s\) is the uniform initial temperature in the medium, \(u_o\) is the moisture content of the surrounding air, \(u_s\) is the uniform initial moisture content in the medium, \(\delta\) is the thermogradient coefficient and \(\varepsilon\) is the phase conversion factor. \(Lu, Pn\) and \(Ko\) denote the Luikov, Posnov and Kossovitch numbers, respectively.

In order to derive the approximate formulations, we integrate equations (5) along the radial direction. By substituting into the resulting expressions the boundary conditions (5.e,f,g,h) and by using the definitions of average temperature and average moisture content at each cross section, \(\bar{\theta}(Z,\tau)\) and \(\bar{\phi}(Z,\tau)\), respectively, we can write the formulation for the coupled heat and mass transfer problem as:

\[
\frac{\partial \bar{\theta}(Z,\tau)}{\partial \tau} = \alpha \frac{\partial^2 \bar{\theta}(Z,\tau)}{\partial Z^2} - \beta \frac{\partial^2 \bar{\phi}(Z,\tau)}{\partial Z^2} + 2r_\alpha^2 Bi_\psi \theta(1,Z,\tau) + 2r_\beta^2 Bi_y, \quad \text{in } 0<Z<1 \text{ for } \tau>0,
\]

\[
\frac{\partial \bar{\phi}(Z,\tau)}{\partial \tau} = Lu \frac{\partial^2 \bar{\phi}(Z,\tau)}{\partial Z^2} - LuPn \frac{\partial^2 \bar{\theta}(Z,\tau)}{\partial Z^2} \quad \text{in } 0<Z<1 \text{ for } \tau=0,
\]

\[
\bar{\theta}(Z,0) = 0, \quad \bar{\phi}(Z,0) = 0.
\]
\[ \frac{\partial \theta(0,\tau)}{\partial Z} = -Q, \quad \frac{\partial \phi(0,\tau)}{\partial Z} = -PnQ, \quad \text{at } Z=0 \text{ for } \tau>0, \quad (7.e,f) \]

\[ \frac{\partial \theta(1,\tau)}{\partial Z} - Bi_q[1- \hat{\theta}(1,\tau)] + (1-\varepsilon)KoLuBi_m[1- \hat{\phi}(1,\tau)] = 0, \quad \text{at } Z=1 \text{ for } \tau>0. \quad (7.g) \]

\[ \frac{\partial \phi(1,\tau)}{\partial Z} + Bi_m^* \hat{\phi}(1,\tau) = Bi_m^* - PnBi_q[\hat{\theta}(1,\tau) - 1], \quad \text{at } Z=1 \text{ for } \tau>0. \quad (7.h) \]

We note in equation (7.a) that such a formulation for the problem involves the temperature at the lateral surface of the body, besides the average temperature and average moisture content at each cross section. Two different approaches are illustrated here to approximate \( \theta(1,Z,\tau) \), as described next.

In the traditional lumped approach, gradients inside the body along the radial direction are neglected. Therefore, we can approximate the temperature at the lateral surface by the average temperature, that is

\[ \theta(1,Z,\tau) = \hat{\theta}(Z,\tau). \quad (8) \]

An improved approximate formulation can be obtained, by using Hermite integrals to write the temperature at the lateral surface of the body in terms of the average temperature at each cross section. In this work, we use the \( H_{1,1} \) expression in order to approximate the average temperature and the \( H_{0,0} \) expression to approximate the integral of the temperature gradient along the radial direction. We then obtain the following expression for the temperature at the lateral surface of the body

\[ \theta(1,Z,\tau) = \frac{4}{4+Bi_{w}} \left[ \hat{\theta}(Z,\tau) + \frac{Bi_w}{4} \right]. \quad (9) \]

By substituting equation (9) into equation (7.a) and rearranging, we obtain a formulation for the present problem similar to that obtained with the classical lumped approach. The only difference is that for the CIEA, a modified Biot number at the lateral surface appears in the formulation. Therefore, the use of the approximation for \( \theta(1,Z,\tau) \) obtained with the CIEA does not introduce any additional complexity into the formulation for the problem, as compared to the lumped approach. However, more accurate results are obtained with the CIEA instead of the lumped approach, since the radial gradients in the body are now accounted for more precisely.

We examine below the effects of the Biot number in the radial direction on the approximate solutions obtained via lumped and \( H_{1,1}/H_{0,0} \) approaches. Figures 1.a,b show the results for \( Bi_{w} = 0, 1.0 \) and 10, as well as the results obtained with the exact one-dimensional solution via GITT [20], for the average temperature and average moisture content at the position \( Z=0 \). Other parameters of importance for the analysis were taken as: \( Lu=0.4, \ Pn=0.6, \ Ko=5.0, \ Bi_q=Bi_m=2.5, \ \varepsilon =0.2 \) and \( Q=0.9 \).

As expected, the lumped and CIEA solutions are in perfect agreement with the 1D solution for \( Bi_{w}=0 \). As \( Bi_{w} \) increases, the average temperatures obtained with the approximate solutions tend to be smaller than that for the 1D solution, due to the lateral heat losses. The same behavior is observed for the average moisture content. By comparing the 2D approximate solutions, we can notice that the average temperatures and the average moisture contents tend to be larger with the \( H_{1,1}/H_{0,0} \) approximation than with the lumped approach. This is due to the fact that the modified Biot number given by eq.(9) for the \( H_{1,1}/H_{0,0} \) approximation, which takes into account the radial gradients, is smaller than the actual Biot number. This effect is more noticeable for larger Biot numbers in the radial direction, such as \( Bi_{w} =10 \). The solution via GITT of the two-dimensional problem given by equations (1) is described in [22], allowing for the verification of the progressive accuracy loss in the classical lumped analysis. The comparison of the two approximate solutions proposed here with such a 2D solution identifies ranges of validity in terms of the radial Biot number, for different values of \( Lu, Pn, Ko, Bi_{w}, Bi_{m}, Q \) and \( \varepsilon \).
HYBRID METHODS: THE GENERALIZED INTEGRAL TRANSFORM TECHNIQUE (GITT)

Discrete numerical methods are still nowadays responsible for most of the challenges and tasks accomplished in engineering analysis, and most frequently employed in commercial software for multipurpose usage. Nevertheless, a number of hybrid methodologies have been appearing in the open literature, which to within different degrees of success, attempt to match the classical analytical ideas with the presently solid knowledge basis on numerical analysis. Within the last two decades, the classical integral transform method [23] gained a hybrid numerical-analytical structure, offering user controlled accuracy and quite efficient computational performance for a wide variety of a priori non transformable problems [24-31], including the nonlinear formulations of interest in heat and fluid flow applications. Besides being an alternative computational method on itself, this hybrid approach is particularly well suited for benchmarking purposes, in light of its automatic error control feature, retaining the same characteristics of a purely analytical solution. In addition to the straightforward error control and estimation, an outstanding aspect of this method is the direct extension to multidimensional situations, with a moderate increase in computational effort with respect to one-dimensional applications. Again, the hybrid nature is responsible for this behavior, since the analytical part in the solution procedure is employed over all but one independent variable, and the numerical task is always reduced to the integration of an ordinary differential system in this one single independent variable.

The present section reviews the concepts behind the Generalized Integral Transform Technique (GITT), as an example of a hybrid method in heat and fluid flow applications, which adds to the available simulation tools, either as a companion in covalidation tasks or as an alternative approach for analytically oriented users. As an illustration of the formal solution procedure, we consider here a transient convection-diffusion problem of coupled potentials (velocity, temperature or concentration), defined in the region $V$ with boundary surface $S$, and including non-linear effects in the convective terms as follows:

$$w_k(x) \frac{\partial T_k(x,t)}{\partial t} + u(x,t,T_r) \nabla T_k(x,t) + LT_k(x,t) = P_k(x,t,T_k),$$

$$x \in V, \ t > 0, \ k, \ell = 1,2,\ldots,n$$

with initial and boundary conditions given, respectively, by

\[ (10.a) \]
\[
T_k(x,0) = f_k(x), \quad x \in V, \quad (10.b)
\]
\[
\left[ \alpha_k(x) + \beta_k(x) k(x) \frac{\partial}{\partial n} \right] T_k(x,t) = \phi_k(x,t), \quad x \in S, \quad t > 0, \quad (10.c)
\]

where the equation operator is written as:
\[
L_k \equiv -\nabla k(x) \nabla + d_k(x)
\]

and \( n \) denotes the outward drawn normal to the surface \( S \).

Without the convection terms and for linear source terms, i.e. \( u(x, t, T_i) \equiv 0, P \equiv P(x, t) \), and \( \phi \equiv \phi(x, t) \), problem (10) becomes a class I linear diffusion problem according to an earlier classification [23], for which exact analytical solutions were obtained through the classical integral transform technique. Otherwise, problem (10) is not a priori transformable, and the ideas in the generalized integral transform technique [24-31] can be utilized to develop hybrid numerical-analytical solutions to this class of problems. Following the formalism previously established for convection-diffusion non-linear problems [25], the appropriate auxiliary problems are taken as:
\[
L_k \psi_{ki}(x) = \mu_{ki}^2 \psi_{ki}(x), \quad x \in V 
\]

with boundary conditions
\[
\left[ \alpha_k(x) + \beta_k(x) k(x) \frac{\partial}{\partial n} \right] \psi_{ki}(x) = 0, \quad x \in S, \quad (11.b)
\]

where the eigenvalues, \( \mu_{ki}^2 \), and related eigenfunctions, \( \psi_{ki}(x) \), are here assumed to be known from application of also recently advanced computational methods for Sturm-Liouville type problems [23, 25]. Problem (11) then allows, through the associated orthogonality property of the eigenfunctions, definition of the integral transform pairs below:
\[
\bar{T}_{ki}(t) = \int_V w_k(x) K_{ki}(x) T_k(x,t) dv, \quad \text{transforms}, \quad (12.a)
\]
\[
T_k(x,t) = \sum_{i=1}^{\infty} K_{ki}(x) \bar{T}_{ki}(t), \quad \text{inverses}, \quad (12.b)
\]

where the symmetric kernels \( K_{ki}(x) \) are given by:
\[
K_{ki}(x) = \frac{\psi_{ki}(x)}{N_{ki}^{1/2}}; \quad N_{ki} = \int_V w_k(x) \psi_{ki}^2(x) dv. \quad (12.c,d)
\]

The integral transformation of (10.a) is now accomplished by applying the operator \( \int_V K_{ki}(x) dv \) to yield, after using boundary conditions (10.c), (11.b):
\[
\frac{d\bar{T}_{ki}(t)}{dt} + \sum_{j=1}^{\infty} a_{kj}(t,T_k) \bar{T}_{kj}(t) = \bar{G}_{ki}(t,T_k), \quad i = 1, 2, ..., \quad t > 0, \quad (13.a)
\]
\[
k_{i, \ell} = 1, 2, ..., n
\]

The initial conditions (10.b) are also integral transformed through the operator \( \int_V w_k(x) K_{ki}(x) dv \), to provide:

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\[
\mathcal{T}_{k,i}(0) = \mathcal{f}_{ki} = \int_{V} w_k(x)K_{k_i}(x)f_k(x)\,dv
\]  

(13.b)

where

\[
\bar{g}_{ki}(t,T_k) = \int_{S} k_i(x)P_i(x,t,T_k)\,dv + \int_{S} k_i(x) \left[ K_{k_i}(x)\frac{\partial T_i(x,t)}{\partial n} - T_k(x,t)\frac{\partial K_{k_i}(x)}{\partial n} \right] \,ds,
\]  

(13.c)

\[
a_{ij}(t,T_i) = \delta_{ij}\mu_k^i_i + a_{ij}^*(t,T_i)
\]  

(13.d)

with

\[
\delta_{ij} = \begin{cases} 
0, & \text{for } i \neq j \\
1, & \text{for } i = j
\end{cases}
\]

(13.e)

\[
a_{ij}^*(t,T_i) = \int_{V} K_{k_i}(x)\left[u(x,t,T_i)\nabla K_{k_j}(x)\right]dv.
\]

(13.f)

Equations (13) form an infinite system of coupled non-linear ordinary differential equations for the transformed potentials, \( \mathcal{T}_{k,i} \)'s. For computation purposes, system (13) is truncated at the Nth row and column, with N sufficiently large for the required convergence. The formal aspects behind the convergence to the infinite system solution as the truncation order, N, is increased, have been investigated in [24, 25]. The non-linear initial value problem defined by eqs. (13) is likely to belong to a class of stiff ordinary differential systems, especially for increasing values of N. Fortunately, various special numerical integrators have been designed within the last two decades, to this class of systems [32, 33]. Once the transformed potentials have been computed from numerical solution of system (13), the inversion formula (12.b) is recalled to reconstruct the original potentials \( \mathcal{T}_k(x,t) \), in explicit form.

Since all the intermediate numerical tasks are accomplished within user prescribed accuracy, one is left with the need of reaching convergence in the eigenfunction expansions and automatically controlling the truncation order \( N \), for the requested accuracy in the final solution. The analytic nature of the inversion formulae allows for a direct testing procedure at each specified position within the medium where a solution is desired, and the truncation order \( N \) can be gradually decreased (or eventually increased), to fit the user global error requirements over all the solution domain. The simple tolerance testing formula employed is written as:

\[
\varepsilon = \max_{x \in D} \left| \frac{\sum_{i=N}^{N} \bar{\psi}_i(x)\mathcal{T}_i(t)}{\mathcal{T}_i(x;t) + \sum_{i=1}^{N} \bar{\psi}_i(x)\mathcal{T}_i(t)} \right|,
\]

(14)

where \( T_i \) is a filtering solution, if eventually employed, and \( N^* \) is decreased from the value of \( N \) while \( \varepsilon \) still fits the user requested global error, and then \( N \) is changed to assume the value of \( N^* \).

A major aspect in the practical implementation of this methodology is the eventual need for improving the convergence behavior of the resulting eigenfunction expansions. One possible alternative for further improvement is the proposition of analytical filtering solutions, which present both space and time dependence, within ranges of the time numerical integration path [3].

In multidimensional applications, the final integral transform solution for the related potential is expressed as double or triple infinite summations for two or three-dimensional transient problems, or a double summation for a three-dimensional steady problem. Each of these summations is associated with the eigenfunction expansion in a corresponding spatial coordinate, eliminated through integral transformation from the partial differential system, and recovered analytically through such expressions. From a computational point of view, only a truncated version of such nested summations

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can be actually evaluated. However, the plain truncation of these series, individually, to a certain prescribed finite order, is certainly not an efficient approach. In this way, some still important information to the final result can be disregarded, while other terms are accounted for that have essentially no contribution to convergence in the relative accuracy required. Therefore, for an efficient computation of these expansions, the infinite multiple summations should first be converted to a single sum representation, with the appropriate reordering of terms, according to their individual contribution to the final numerical result [3]. Then, one would be able to evaluate a minimum number of eigenvalues and related derived quantities, as many as required for reaching the user prescribed accuracy target.

Application: Drying in Capillary Porous Media (GITT)

The potential of the integral transform approach in dealing with different classes of linear or nonlinear problems in heat and mass transfer is now briefly illustrated. The example selected again involves simultaneous heat and mass transfer during drying of a capillary porous body under the Luikov model, according to the one-dimensional problem formulation previously handled in [18-20]. The ideas above discussed were directly applied to both linear and nonlinear formulations in [18, 19], and such analysis may be rederived by following the above steps with the appropriate correspondence between the original drying problem and the fairly general formulation within eqs.(10). Due to space limitations, and so as to complement in scope the above analysis, we have chosen to briefly demonstrate the alternative GITT path for handling the Luikov system of equations by employing the integral transform approach directly on the eigenvalue problem that provides the exact integral transformation [20]. Additional details on the application of the method can be readily obtained from the original papers that correspond to these topics, and/or from the recent compilations in [28-31], where several other topics not presented here are also discussed.

The system of equations originally proposed by Luikov [16, 17] for modeling drying of capillary porous media, involving temperature and moisture potential distributions in a slab is here considered for illustration. This is the same drying problem solved in refs.[23, 34] without consideration of the associated complex eigenvalues, and later analyzed by Lobo et al. [35] with the inclusion of one pair of conjugate roots of the transcendental equation. Therefore, the present alternative solution is demonstrated for this same one-dimensional linear problem to allow for critical and definitive comparisons. The problem formulation in dimensionless form is given by [23]:

\[
\frac{\partial \theta_1(X,\tau)}{\partial \tau} = \frac{\partial^2 \theta_1(X,\tau)}{\partial X^2} - \varepsilon Ko \frac{\partial \theta_2(X,\tau)}{\partial \tau} \quad \text{in } 0 \leq X \leq 1, \tau > 0, \quad (15.a)
\]

\[
\frac{\partial \theta_2(X,\tau)}{\partial \tau} = Lu \frac{\partial^2 \theta_1(X,\tau)}{\partial X^2} - Lu Pn \frac{\partial^2 \theta_1(X,\tau)}{\partial X^2} \quad \text{in } 0 \leq X \leq 1, \tau > 0, \quad (15.b)
\]

subject to the initial conditions

\[
\theta_1(X,0) = 0, \quad \theta_2(X,0) = 0, \quad \text{in } 0 \leq X \leq 1, \quad (15.c,d)
\]

and boundary conditions given by

\[
\frac{\partial \theta_1(0,\tau)}{\partial X} = - Q; \quad \frac{\partial \theta_2(0,\tau)}{\partial X} - Pn \frac{\partial \theta_1(0,\tau)}{\partial X} = 0, \quad \tau > 0, \quad (15.e,f)
\]

\[
\frac{\partial \theta_1(1,\tau)}{\partial X} - B_i(1 - \theta_1(1,\tau)) \frac{\partial \theta_1(1,\tau)}{\partial X} - KoLu \frac{\partial \theta_2(1,\tau)}{\partial X} = 0, \quad \tau > 0, \quad (15.g)
\]

\[
\frac{\partial \theta_2(1,\tau)}{\partial X} - Pn \frac{\partial \theta_1(1,\tau)}{\partial X} + Bi \frac{\partial \theta_2(1,\tau)}{\partial X} = 0, \quad \tau > 0, \quad (15.h)
\]

where the various dimensionless groups are defined as previously indicated.
Mikhailov & Özisik [23] developed the exact solution of eqs. (15) using the Classical Integral Transform Technique, and they obtained the following final expressions:

\[ \theta_1 (X, \tau) = 1 + Q \left[ 1 + \frac{1}{B_{i_q}} - X \right] - \sum_{i=1}^{\infty} A_i \exp(-\mu_i^2 \tau) \left( \sum_{k=1}^{3} (-1)^k (v_i^2 - 1) b_i (\mu_i) \cos(v_{3,4} \mu_i X) \right) , \]  

(16.a)

\[ \theta_2 (X, \tau) = 1 + PnQ (1 - X) + Pn \sum_{i=1}^{\infty} A_i \exp(-\mu_i^2 \tau) \left( \sum_{k=1}^{3} (-1)^k b_i (\mu_i) \cos(v_{3,4} \mu_i X) \right) \]  

(16.b)

with

\[ v_i^2 = \frac{1}{2} \left[ 1 + \varepsilon PnKo + \frac{1}{Lu} \right] + (-1)^k \left[ \left( 1 + \varepsilon PnKo + \frac{1}{Lu} \right)^2 - \frac{4}{Lu} \right] , \quad k = 1, 2 , \]  

(16.c,d)

\[ a_i (\mu_i) = \left[ 1 + (1 - v_i^2) \frac{1 - \varepsilon}{\varepsilon} \frac{Lu B_{i_m}}{Bi_q} \right] \cos(v_{3,4} \mu_i) - \frac{v_{3,4} \mu_i}{Bi_q} \sin(v_{3,4} \mu_i) , \quad k = 1, 2 , \]  

(16.e,f)

\[ b_i (\mu_i) = \cos(v_{3,4} \mu_i) - \frac{\mu_i}{Lu B_{i_m} v_i} \sin(v_{3,4} \mu_i) , \quad k = 1, 2 , \]  

(16.g,h)

\[ e_i = (v_i^2 - 1) \frac{1}{Bi_q} - \frac{1}{Lu B_{i_m}} - KoPn \frac{B_{i_m}}{Bi_q} , \quad d_i (\mu_i) = 1 + \cos(v_{3,4} \mu_i) \frac{\sin(v_{3,4} \mu_i)}{v_{3,4} \mu_i} , \quad k = 1, 2 , \]  

(16.i-l)

\[ A_i = \frac{2}{\mu_i^2} \left[ \frac{Q}{v_i^2 - v_{3,4}^2} \left[ b_i (\mu_i) e_2 - b_i (\mu_i) e_1 \right] + \left[ \left( 1 - \varepsilon \right) KoLu \frac{B_{i_m}}{Bi_q} \right] b_i (\mu_i) + \frac{v_{3,4}^2 - 1}{Pn} a_i (\mu_i) \right] b_i (\mu_i) \]  

\[ \times \left[ b_i^2 (\mu_i) e_2 e_1 (\mu_i) - b_i^2 (\mu_i) e_1 d_2 (\mu_i) \right] \]  

(16.m)

The numbers \( \mu_i \) are solutions of the eigenvalue problem:

\[ \psi^*_i (X) + \mu_i v_i^2 \psi_i (X) = 0 , \quad 0 < X < 1 , \quad k = 1, 2 , \]  

(17.a,b)

\[ \psi_i (0) = 0 , \quad k = 1, 2 \]  

(17.c,d)

\[ \sum_{k=1}^{3} (-1)^k \left[ 1 - v_{3,4}^2 \right] \left[ \frac{\mu_i^2}{\varepsilon} \frac{Lu B_{i_m}}{Bi_q} \right] \psi_i (1) + \frac{1}{Bi_q} \psi^*_i (1) = 0 , \]  

(17.e)

\[ \sum_{k=1}^{3} (-1)^k \left[ \psi_i (1) + \frac{v_{3,4}^2 - 1}{Bi_m} \psi^*_i (1) \right] = 0 . \]  

(17.f)

The solution of equations (17.a,b) is taken in the form:

\[ \psi_i (\mu X) = C_i \cos(v_i \mu X) + D_i \sin(v_i \mu X) , \]  

(18.a,b)

where \( C_i \) and \( D_i \) are constants to be determined by utilizing the boundary conditions for the problem.

From the boundary conditions (4c,d) it follows that
$$D_k = 0,$$  \hspace{2cm} (18.c)

The substitution of the solution (18) into the boundary conditions (17.e) and (17.f), recalling that
$$\nu^2_{k}\nu^2_{3-k} = 1/Lu,$$ supplies:

$$\sum_{k=1}^{2} (-1)^k (1 - \nu^2_{k}) a_k (\mu) C_k = 0, \hspace{1cm} \sum_{k=1}^{2} (-1)^k b_k (\mu) C_k = 0. \hspace{1cm} (19), (20)$$

The eigencondition for the determination of the eigenvalues $\mu_i$ is obtained as:

$$\sum_{k=1}^{2} (-1)^k (1 - \nu^2_{k}) a_{3-k} (\mu) b_k (\mu) = 0,$$  \hspace{2cm} (21)

where $a_k (\mu)$ and $b_k (\mu)$ were already defined in (16.e-h). Mikhailov & Özisik [23] tried to solve this
transcendental equation directly, but the classic methods for this nonlinear equation presented two
types of difficulty:

- some real roots could be missed,
- all the possible complex roots were omitted.

These are the reasons while it was here preferred to choose another procedure to calculate the
eigenvalues of eqs.(17.a,b), once the classic procedures were already erroneously employed in
previous works.

To apply the GITT to the eigenvalue problem, eqs. (17.a,b), it was initially chosen an auxiliary
problem of well-known solution, which retains as much as possible, pertinent information about the
original problem. This auxiliary problem will then supply the basis for the expansion of the original
eigenvalue problem.

The equations to be solved, coupled at the boundaries, are now rewritten as:

$$\psi_i' (X) + \mu^2 \nu^2_i \psi_i (X; \mu) = 0, \hspace{1cm} 0 < X < 1, \hspace{2cm} (22.a)$$

$$\psi_i' (X) + \mu^2 \nu^2_i \psi_i (X; \mu) = 0, \hspace{1cm} 0 < X < 1, \hspace{2cm} (22.b)$$

subject to the boundary conditions,

$$\psi_i (0) = 0, \hspace{1cm} \psi_i (0) = 0, \hspace{2cm} (22.c)$$

$$\alpha_{i1} \psi_i (1) + \alpha_{i2} \psi_i (1) + \beta_{i1} \psi_i (1) + \beta_{i2} \psi_i (1) = 0, \hspace{2cm} (22.d)$$

$$\alpha_{i1} \psi_i (1) + \alpha_{i2} \psi_i (1) + \beta_{i1} \psi_i (1) + \beta_{i2} \psi_i (1) = 0. \hspace{2cm} (22.e)$$

The system above characterizes a typical eigenvalue problem associated with the so-called Class
III diffusion problems in [23], which is here intended to be solved through generalized integral
transformation. Following the formalism in the GITT [25-31], two independent auxiliary problems of
the Sturm-Liouville type, respectively, for the temperature and moisture expansions, are now chosen.

Temperature:

$$\chi_{1,i} (x) + \lambda^2_{1,i} \nu^2_{1,i} \chi_{1,i} (x) = 0, \hspace{2cm} (23.a)$$

$$\chi_{1,i} (0) = 0, \hspace{1cm} \alpha_{11} \chi_{1,i} (1) + \beta_{1j} \chi_{1,i} (1) = 0. \hspace{2cm} (23.b,c)$$

Moisture:

$$\chi_{2,i} (x) + \lambda^2_{2,i} \nu^2_{2,i} \chi_{2,i} (x) = 0, \hspace{2cm} (24.a)$$

$$\chi_{2,i} (0) = 0, \hspace{1cm} \alpha_{22} \chi_{2,i} (1) + \beta_{2j} \chi_{2,i} (1) = 0. \hspace{2cm} (24.b,c)$$
The two auxiliary problems above will serve as the basis for the eigenfunction expansions, of the unknown eigenfunctions of the original problem (22).

The auxiliary problems (23), (24) allow the definition of the following integral transform pairs for the potentials \( \psi_1 \) and \( \psi_2 \)

Temperature:

\[
\psi_1(x, \mu) = \sum_{i=1}^{\infty} \frac{1}{M_{i,i}^{X}} \chi_{1,i}(x) \tilde{\psi}_{1,i} \quad \text{inverse} \tag{25.a}
\]

\[
\tilde{\psi}_{1,i} = \frac{1}{M_{i,i}^{X}} \int_{0}^{1} v_1^2 \chi_{1,i}(x) \psi_1(x, \mu) \, dx \quad \text{transform} \tag{25.b}
\]

and \( M_{1,i} = \int_{0}^{1} v_1^2 \chi_{1,i}^2(x) \, dx \). \tag{25.c}

Moisture:

\[
\psi_2(x, \mu) = \sum_{i=1}^{\infty} \frac{1}{M_{2,i}^{X}} \chi_{2,i}(x) \tilde{\psi}_{2,i} \quad \text{inverse} \tag{26.a}
\]

\[
\tilde{\psi}_{2,i} = \frac{1}{M_{2,i}^{X}} \int_{0}^{1} v_2^2 \chi_{2,i}(x) \psi_2(x, \mu) \, dx \quad \text{transform} \tag{26.b}
\]

and \( M_{2,i} = \int_{0}^{1} v_2^2 \chi_{2,i}^2(x) \, dx \), \tag{26.c}

where \( M_{1,i} \) and \( M_{2,i} \) are the normalization integrals.

The solutions of the auxiliary problems are readily obtainable as:

\[
\chi_{1,i}(X) = \cos(\lambda_{1,i} v_1 X), \quad (\lambda_{1,i} v_1) \tan(\lambda_{1,i} v_1) = \frac{\alpha_{11}}{\beta_{11}}, \quad M_{1,i} = v_1^2 \left[ \frac{1}{2} + \frac{\sin(2\lambda_{1,i} v_1)}{4\lambda_{1,i} v_1} \right], \tag{27.a-c}
\]

\[
\chi_{2,i}(X) = \cos(\lambda_{2,i} v_2 X), \quad (\lambda_{2,i} v_2) \tan(\lambda_{2,i} v_2) = \frac{\alpha_{22}}{\beta_{22}}, \quad M_{2,i} = v_2^2 \left[ \frac{1}{2} + \frac{\sin(2\lambda_{2,i} v_2)}{4\lambda_{2,i} v_2} \right]. \tag{28.a-c}
\]

The next step is then to perform the integral transformation of the original eigenvalue problem, in order to reduce it into an algebraic system (matrix eigensystem analysis). For this purpose the operator \( \int_{0}^{1} \chi_{1,i}(X) \, dX \) is applied on eq. (22.a) and the operator \( \int_{0}^{1} \chi_{2,i}(X) \, dX \) is applied on eq. (22.b). Through manipulations, including integration by parts and accounting for the boundary conditions, we meet expressions that relate the transformed potentials of the original and auxiliary problems, in the following way [20]:

\[
(\mu^2 - \lambda_{1,i}^2) \tilde{\psi}_{1,i} = \frac{\chi_{1,i}(1)}{M_{1,i}^{X}} \left[ \frac{1}{\beta_{1i}} \left( \alpha_{11} \psi_{1}(1) + \beta_{1i} \psi_{2}(1) \right) \right] = 0, \tag{29.a}
\]

\[
(\mu^2 - \lambda_{2,i}^2) \tilde{\psi}_{2,i} = \frac{\chi_{2,i}(1)}{M_{2,i}^{X}} \left[ \frac{1}{\beta_{2i}} \left( \alpha_{11} \psi_{1}(1) + \beta_{2i} \psi_{2}(1) \right) \right] = 0. \tag{29.b}
\]
We now make use of integral balance equations [25] to obtain expressions for \( \psi_1'(1) \) and \( \psi_2'(1) \), in order to accelerate the convergence of the eigenfunction expansions, especially in the vicinity of the boundaries. Integrating eqs. (22.a,b) between 0 and 1, and employing the boundary conditions:

\[
\psi_1'(1) = -\mu^2 v_1^2 \sum_{j=1}^{\infty} \left[ \frac{1}{M_{1,j}} \int_0^1 \varphi_{1,j} (X) \, dX \right], \quad \psi_2'(1) = -\mu^2 v_2^2 \sum_{j=1}^{\infty} \left[ \frac{1}{M_{2,j}} \int_0^1 \varphi_{2,j} (X) \, dX \right],
\]

The direct substitution of the inverse formulas (25.a) and (26.a) at the boundaries expressions is not recommended [18-20], since the boundary conditions of the auxiliary problems do not exactly correspond to those of the original problem. Then, expressions for \( \psi_1'(1) \) and \( \psi_2'(1) \) in terms of the derivatives, already obtained in the previous step, are then readily derived as:

\[
\psi_1'(1) = \gamma_1 \psi_1'(1) + \gamma_2 \psi_2'(1),
\]

\[
\gamma_1 = \frac{\alpha_{12} \beta_{21} - \alpha_{21} \beta_{12}}{\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}} \quad \text{and} \quad \gamma_2 = \frac{\alpha_{12} \beta_{22} - \alpha_{22} \beta_{12}}{\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}},
\]

\[
\psi_2'(1) = \gamma_1 \psi_1'(1) + \gamma_3 \psi_2'(1),
\]

\[
\gamma_3 = \frac{\alpha_{11} \beta_{21} - \alpha_{21} \beta_{11}}{\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}} \quad \text{and} \quad \gamma_3 = \frac{\alpha_{11} \beta_{22} - \alpha_{21} \beta_{12}}{\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}}.
\]

From substitution of the above relations in the transformed equations, we obtain:

\[
(\mu^2 - \lambda_{1,j}^2) \varphi_{1,j} + \mu^2 \sum_{j=1}^{\infty} A_{j,j} \varphi_{1,j} + \mu^2 \sum_{j=1}^{\infty} B_{j,j} \varphi_{2,j} = 0,
\]

where

\[
A_{i,j} = \frac{1}{\beta_{11}} \alpha_{12}\gamma_j v_1^2 \chi_{1,j}(1) \frac{1}{M_{1,j}} \int_0^1 \chi_{1,j}(x) \, dx,
\]

\[
B_{i,j} = \frac{1}{\beta_{11}} (\alpha_{12} \gamma_2 + \beta_{21}) v_2^2 \chi_{2,j}(1) \frac{1}{M_{1,j}} \int_0^1 \chi_{2,j}(x) \, dx,
\]

\[
(\mu^2 - \lambda_{2,j}^2) \varphi_{2,j} + \mu^2 \sum_{j=1}^{\infty} C_{i,j} \varphi_{1,j} + \mu^2 \sum_{j=1}^{\infty} D_{i,j} \varphi_{2,j} = 0,
\]

where

\[
C_{i,j} = \frac{1}{\beta_{22}} (\alpha_{21} \gamma_1 + \beta_{21}) v_1^2 \chi_{2,j}(1) \frac{1}{M_{2,j}} \int_0^1 \chi_{1,j}(X) \, dX,
\]

\[
D_{i,j} = \frac{1}{\beta_{22}} \alpha_{21} \gamma_2 v_2^2 \chi_{2,j}(1) \frac{1}{M_{2,j}} \int_0^1 \chi_{2,j}(X) \, dX.
\]

The generalized boundary conditions used in the original problem, eqs. (22.d, e), after being compared to those presented by Mikhailov & Özisik [23] in the specific drying problem, eqs. (15), allows for the correspondence with the example considered, in the form:

\[
\alpha_{11} = -1, \quad \alpha_{12} = 1, \quad \beta_{11} = -\frac{v_1^2}{B_{i_1}}, \quad \beta_{12} = \frac{v_2^2}{B_{i_2}},
\]

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After truncation of the infinite systems (33) and (34) in a sufficiently large finite order, \( N \), for the desired precision, the resulting finite system is written in matrix form as:

\[
\begin{pmatrix}
\hat{\lambda}_{i,j}^2 \delta_{i,j} & 0 \\
0 & \hat{\lambda}_{j,i}^2 \delta_{i,j}
\end{pmatrix}_{2N \times 2N}
\begin{bmatrix}
\varphi_i \\
\varphi_j
\end{bmatrix}
= 
\begin{bmatrix}
A_{i,j} + \delta_{i,j} \\
C_{i,j} + \delta_{i,j}
\end{bmatrix}_{2N \times 2N}
\begin{bmatrix}
\varphi_i \\
\varphi_j
\end{bmatrix}
\]

or

\[
[A]\{\varphi\} = \nu \begin{bmatrix}
A \\
B
\end{bmatrix}\{\varphi\}.
\]

where \( \nu = \mu^2 \Rightarrow \mu = \sqrt{\nu} \),

\[
\{\varphi\} = \{\varphi_{1,1}, \varphi_{1,2}, \ldots, \varphi_{1,N}, \varphi_{2,1}, \varphi_{2,2}, \ldots, \varphi_{2,N}\}^T.
\]

Using standard algorithms from scientific subroutine libraries [32], the algebraic eigenvalue problem (36) is numerically solved for the eigenvalues and eigenvectors. In this subroutine the user supplies the matrices \( A \) and \( B \), then the desired \( 2N \) eigenvalues are computed with automatic accuracy control. For comparison purposes, the same problem was calculated with the use of the software Mathematica [33], with excellent agreement. Finally, the dimensionless temperature and moisture profiles are computed from eqs. (16.a,b).

Table 1 and 2 illustrate the convergence of the eigenvalues calculations for one specific case (\( Lu = 0.4, \ Pn = 0.6, \ \varepsilon = 0.2, \ \text{Ko}=5.0, \ B_{L,v} = B_{v} = 2.5, \ Q=0.9 \)), respectively, for the first ten eigenvalues and for all of the complex eigenvalues found up to the maximum truncation order employed, demonstrating the excellent convergence behavior of the proposed eigenfunction expansions of the original eigenvalue problem. The present approach once more demonstrated that the eigenvalue tables published in the literature [34] omit some real eigenvalues and all the complex eigenvalues associated to the exact solution of the Luikov equations; therefore the temperature and moisture potential distributions presented in such reference are incorrect, as previously pointed out in [35] by the same authors. Figures 2 illustrate the effects of omitting these eigenvalues, as evident from the erroneous time evolutions for the space averaged temperature and moisture distributions achieved in specific cases.

CONCLUSION

The use of hybrid tools in formulation, solution and computation of thermal problems has been discussed and illustrated. The hybrid nature present in these research fronts has been allowing for exciting findings on improved characteristics and for continuous progress in comparison with conventional approaches. While much has already been achieved, as demonstrated by the ample literature available, research needs, at the same pace, become more evident. The Coupled Integral Equations Approach has been recently employed to provide \textit{a priori} error analysis, with encouraging results, and should be progressively extended to more complex nonlinear formulations. The Generalized Integral Transform Technique enters now a phase of algorithm refinement and optimization, which includes advanced filtering and reordering schemes, enhanced approaches for ODE systems, and automatic implementation for arbitrarily irregular geometries. The Mathematica system has been intensively employed in conjunction with the approaches above described, aimed at further facilitating the analytical development task.

<table>
<thead>
<tr>
<th>Table 1.</th>
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<tr>
<td>Convergence behavior of the first ten eigenvalues</td>
</tr>
<tr>
<td>(( Lu = 0.4, \ Pn = 0.6, \ \varepsilon = 0.2, \ \text{Ko}=5.0, \ B_{L,v} = B_{v} = 2.5, \ Q=0.9 )</td>
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</table>
The numerical modeling of drying in capillary porous media requires the accurate knowledge of several thermophysical and boundary condition parameters that appear in the formulation. The use of inverse analysis techniques permits the estimation of several of such parameters, from the knowledge of temperature and moisture content measurements taken in the media. The direct problem solution paths above discussed were also employed in different inverse problem analysis for drying of capillary porous media [36-40].

Acknowledgements

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Fig. 2. Influence of the inclusion of complex eigenvalues and of Biot number for mass transfer on the average dimensionless temperature and moisture $Lu = 0.4$, $Ph = 0.6$, $\varepsilon = 0.2$, $Ko=5.0$, $Bi_m = Bi_t = 2.5$, $Q=0.9$)

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